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# On the relation between effective supersymmetric actions in different dimensions.

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## Abstract

We make two remarks: *(i)* Renormalization of the effective charge in a 4-dimensional (supersymmetric) gauge theory is determined by the same graphs and is rigidly connected to the renormalization of the metric on the moduli space of the classical vacua of the corresponding reduced quantum mechanical system. Supersymmetry provides constraints for possible modifications of the metric, and this gives us a simple proof of nonrenormalization theorems for the original 4-dimensional theory. *(ii)* We establish a nontrivial relationship between the effective  $(0+1)$ -dimensional and  $(1+1)$ -dimensional Lagrangians. (The latter represent conventional Kählerian  $\sigma$  models.)

# 1 Introduction.

Consider 4-dimensional supersymmetric gauge theory placed in a small spatial torus  $T^3$  of size  $L$ . We assume that  $g^2(L) \ll 1$  and perturbation theory makes sense. For unitary and symplectic gauge groups  $G$ , the only classical vacua of this theory are given by constant gauge potentials  $A_k$ ,  $k = 1, 2, 3$ , lying in the Cartan subalgebra of the group<sup>1</sup>  $G$ . The low-energy dynamics of the model is determined by the effective Hamiltonian describing motion over the vacuum moduli space. Due to supersymmetry, the energy of a classical vacuum configuration stays zero after loop corrections are taken into account – no potential is generated.<sup>2</sup> However, supersymmetry usually allows the existence of a nontrivial metric on the moduli space, in which case such a metric is generated after loop corrections are taken into account.

The loop corrections to the effective Hamiltonian were calculated first in Ref.[3] in the simplest case of  $\mathcal{N} = 1$  supersymmetric QED with two chiral matter multiplets of opposite charges. In this case, the moduli space is represented by the constant gauge potentials  $A_k$  and their superpartners. Note that, for a field theory on  $T^3$ , the moduli space is compact,  $0 \leq A_k \leq 2\pi/L$ .

The original calculation was carried out in the Hamiltonian framework. The effective Hamiltonian is expressed in terms of  $A_k$ ,  $P_k = -i\partial/\partial A_k$ , and the zero Fourier mode of the photino field  $\psi_\alpha$ ,  $\alpha = 1, 2$ . It has the form

$$\begin{aligned} \frac{1}{e^2} H^{\text{eff}} = & \frac{1}{2} f(\mathbf{A}) P_k^2 f(\mathbf{A}) - \\ & - \epsilon_{j k p} \bar{\psi} \sigma_j \psi f(\mathbf{A}) \partial_p f(\mathbf{A}) P_k - \frac{1}{2} f(\mathbf{A}) \partial_k^2 f(\mathbf{A}) (\bar{\psi} \psi)^2, \end{aligned} \quad (1.1)$$

where  $\sigma_i$  are the Pauli matrices and

$$f(\mathbf{A}) = 1 - \frac{e^2}{4} \sum_{\mathbf{n}} \frac{1}{|\mathbf{A}L - 2\pi\mathbf{n}|^3} + \dots \quad (1.2)$$

(we have rescaled  $A \rightarrow A/e$  compared with the normalization of Ref. [3]). The dots stand for possible higher-loop corrections. The expression (1.2) is written for the theory where the charged fields are massless.

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<sup>1</sup>It is not the case for higher orthogonal and exceptional groups [1], but these complications are beyond the scope of the present paper.

<sup>2</sup>This is true only for nonchiral theories, which are only considered in the present paper. In the theories with chiral matter content, the situation is more complicated [2].

Note that the sum in the right side of Eq. (1.2) diverges logarithmically at large  $|\mathbf{n}|$ . This is none other than the effective charge renormalization

$$e^2(L) = e_0^2 \left[ 1 - \frac{e_0^2}{4\pi^2} \ln(\Lambda_{UV} L) + \dots \right] . \quad (1.3)$$

In the massive case and if the box is large enough,  $\ln(\Lambda L)$  is substituted by  $\ln(\Lambda/m)$ . On the other hand, if we are dealing with dimensionally reduced SQED, where all Fourier harmonics with  $\mathbf{n} \neq 0$  are ignored, we obtain

$$f(\mathbf{A}) = 1 - \frac{e^2}{4|L \mathbf{A}|^3} . \quad (1.4)$$

It is obvious that the coefficients in Eq. (1.4) and in Eq. (1.3) are related<sup>3</sup>. The knowledge of  $\beta$ -function allows one to determine the modification of the metric on the moduli space, and this is how the effective Hamiltonian for  $\mathcal{N} = 1$  non-Abelian theories was evaluated in recent Ref. [5]. The inverse is also true, however, and this is one of the main emphasize of the present paper. We note that the  $\beta$ -function of field theories can be conveniently calculated via modification of the metric in the quantum mechanical limit where all nonzero Fourier harmonics are ignored. Ideologically, this is the ultraviolet cutoff procedure brought to its extreme. One can call it ultraviolet *chopoff*.

As was mentioned, the result (1.1) was first obtained in the Hamiltonian framework using a systematic Born–Oppenheimer expansion for  $H^{\text{eff}}$ . There is, however, a simpler way to derive the same result: to evaluate the term  $\propto \dot{\mathbf{A}}\dot{\mathbf{A}}$  in the effective *Lagrangian* in a slowly varying bosonic background  $\mathbf{A}(t), \psi_\alpha = 0$ . Other structures in the Lagrangian can be restored using supersymmetry. The plan of the paper is the following. In Sect. 2 we present one-loop calculations of the effective Lagrangian. We use the background field method and demonstrate that the result is given by exactly the same graphs as the graphs determining the 4-dimensional  $\beta$ -function. Like in

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<sup>3</sup>The procedure for getting the aforementioned relation is similar to the T-duality transformation on D-branes in string theory (see in particular Ref. [4]). In fact,  $N = 2$  SQED with doublet of hypermultiplets (plus free uncharged hypermultiplets) reduced to one dimension can be interpreted as the theory of a D0-brane in the vicinity of a D4-brane. The latter system was extensively studied [4]. In this note we concentrate, however, on the  $N = 1, d = 4$  systems, which were not considered so far in the string theory framework.

4-dimensional case, the contribution of scalar determinant cancels out in supersymmetric case, and we are left with the graphs describing fermion and gauge boson magnetic interactions.

Next, we go beyond one loop and prove in Sect. 3 non-renormalization theorems for four-dimensional  $\mathcal{N} = 2$  and  $\mathcal{N} = 4$  SYM theories. Sect. 4 is not devoted to the  $\beta$  function, but addresses a related question of the connections between effective models in different dimensions. The Hamiltonian (1.1) represents a nonstandard  $\sigma$  model. The model enjoys  $\mathcal{N} = 2$  QM symmetry (it has 2 different complex supercharges), but is not a Kählerian model (Kählerian  $\sigma$  models are defined on even-dimensional target spaces, whereas the model (1.1) is defined on a 3-dimensional conformally flat manifold). We show that the model (1.1) is related, however, to Kählerian models in a nontrivial way: to obtain a Kählerian model out of Eq. (1.1), one has to go back to the original 4-dimensional field theory and consider it on an asymmetric torus, with one of the sizes much larger than the others.

## 2 Ultraviolet chopoff and $\beta$ function.

Let us consider for definiteness  $\mathcal{N} = 1$  four-dimensional  $SU(2)$  SYM theory

$$\mathcal{L} = \frac{1}{g^2} \text{Tr} \left( -\frac{1}{2} F_{\mu\nu}^2 + i \bar{\lambda} \not{D} \lambda \right), \quad (2.5)$$

where  $\lambda$  are Majorana fermions in the adjoint representation of  $SU(2)$  and  $D_\mu = \partial_\mu - i[A_\mu, \cdot]$ . Put the system in a small spatial box and impose the periodic boundary conditions. We would like to calculate quantum corrections to the effective action in the abelian background field  $A_\mu = C_\mu t^3$ ,  $C_\mu = (0, C_i)$ . We assume that  $C_i$  varies slowly with time, but does not depend on spatial coordinates<sup>4</sup>. The background fermionic fields (superpartners of  $C_\mu$ ) are taken to be zero at this stage.

The calculation can be conveniently done using background gauge method [8]. We decompose the field in the classical (abelian) background and quan-

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<sup>4</sup>The setup of the problem is basically the same as in Refs.[6],[7]. There are two differences: (i) We are considering the  $\mathcal{N} = 1$  rather than  $\mathcal{N} = 4$  theory and the corrections are not going to vanish. (ii) The authors of Ref. [6] did their calculation bearing in mind the geometric picture of scattered  $D0$  branes and their background was slightly more sophisticated than ours.

tum fluctuations,

$$A_\mu \rightarrow C_\mu t^3 + \mathcal{A}_\mu , \quad (2.6)$$

and add to the Lagrangian the gauge-fixing term

$$- \frac{1}{2g^2} (D_\mu^{\text{cl}} \mathcal{A}_\mu)^2 , \quad (2.7)$$

where  $D_\mu^{\text{cl}} = \partial_\mu - i [A_\mu^{\text{cl}}, \cdot]$ . In what follows we use the notation  $A_\mu \equiv A_\mu^{\text{cl}} = C_\mu t^3$ . The coefficient chosen in Eq. (2.7) defines the ‘‘Feynman background gauge’’, which is simpler and more convenient than others. Adding (2.7) to the first term in Eq. (2.5) and integrating by parts, we obtain for the gauge-field-dependent part of the Lagrangian

$$\mathcal{L}_\mathcal{A} = -\frac{1}{2g^2} \text{Tr} (F_{\mu\nu}^2) + \frac{1}{g^2} \text{Tr} \left\{ \mathcal{A}_\mu \left( D^2 g_{\mu\nu} \mathcal{A}_\nu - 2i [F_{\mu\nu}, \mathcal{A}_\nu] \right) \right\} + \dots , \quad (2.8)$$

where the dots stand for the terms of higher order in  $\mathcal{A}_\mu$ . The ghost part of the Lagrangian is

$$\mathcal{L}_{ghost} = -2 \text{Tr} (\bar{c} D^2 c) + \text{higher order terms} \quad (2.9)$$

Now we can integrate over the quantum fields  $\mathcal{A}_\mu$ ,  $c$ , and also over the fermions using the relation

$$(i\not{D})^2 = -D^2 + \frac{i}{2} \sigma_{\mu\nu} F_{\mu\nu} ,$$

$\sigma_{\mu\nu} = \frac{1}{2} [\gamma_\mu, \gamma_\nu]$ . We obtain the effective action as follows:

$$S_{\text{eff}} = -\frac{1}{2g^2} \int_{T^3 \times R} \text{Tr}(F_{\mu\nu}^2) + \log \left( \frac{\det^{\frac{1}{4}} \left( -D^2 I + \frac{i}{2} \sigma_{\mu\nu} [F_{\mu\nu}, \cdot] \right) \det(-D^2)}{\det^{\frac{1}{2}} (-D^2 g_{\mu\nu} + 2i [F_{\mu\nu}, \cdot])} \right) . \quad (2.10)$$

We see that the fermion and gauge field determinants involve, besides the term  $-D^2$  which is present also in the scalar determinant, the term  $\propto F_{\mu\nu}$  describing the magnetic moment interactions. An important observation is

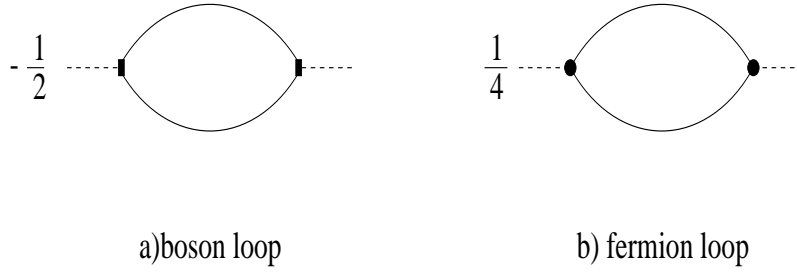


Figure 1: One-loop renormalization of the kinetic term in SYM. Internal lines are Green's functions of the operator  $(-D^2)$ . The vertices involve the spin operator  $J_{\alpha\beta}$  and are different for the fermion and gauge boson loop.

that, were these magnetic interactions absent, the contributions of the ghosts, fermions, and gauge bosons would just cancel and the effective action would not acquire any corrections. This feature is common for all supersymmetric gauge theories ( $\mathcal{N} = 1$ ,  $\mathcal{N} = 2$ ,  $\mathcal{N} = 4$ ; non-Abelian and Abelian). This fact is related to another known fact that, when supersymmetric  $\beta$  function is calculated in the *instanton* background, only the contribution of the zero modes survives [9].

For nonsupersymmetric theories, also nonzero instanton modes provide a nonvanishing contribution in the  $\beta$  function. On the other hand, the contributions due to  $\det(-D^2)$  in the effective action do not vanish in the nonsupersymmetric case.

To find the magnetic contributions, we have to calculate the graphs drawn in Fig. 1. The vertices there are proportional to  $\epsilon^{abc} F_{\alpha\beta} J_{\alpha\beta}$  ( $J_{\alpha\beta}$  being the spin operator in the corresponding representation) and the lines are Green's functions of the operator  $-D^2$ . Only the color components 1 and 2 circulate in the loops. They acquire the mass  $|\mathbf{C}|$  in the Abelian background  $C_i t^3$ . One can be convinced that gauge boson loop involves the factor  $-4$  compared to the fermion one [the factor  $\frac{1}{2} : -\frac{1}{4} = -2$  is displayed in Eq. (2.10) and Fig. 1 and another factor 2 comes from spin; see Eq. (16.128) in Peskin's book].

Let us calculate, say, the fermion loop. If all higher Fourier modes are “chopped off”,  $-D^2 \rightarrow -\partial_0^2 - \mathbf{C}^2$  and the corresponding contribution to the effective Lagrangian is

$$\frac{1}{4} \cdot \frac{1}{2} \cdot 2 \cdot \dot{C}_j \dot{C}_k \text{Tr}\{\sigma_{0j} \sigma_{0k}\} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{(\omega^2 + \mathbf{C}^2)^2} = \frac{\dot{\mathbf{C}}^2}{4|\mathbf{C}|^3}. \quad (2.11)$$

(the factor  $\frac{1}{4}$  is the power of the determinant in Eq. (2.10),  $\frac{1}{2}$  comes from the expansion of the logarithm and 2 is the color factor.) Adding the gauge boson contribution and also the free bosonic term, we obtain

$$\frac{g^2}{L^3} \mathcal{L}_{\text{bos}}^{\text{eff}} = \frac{\dot{\mathbf{C}}^2}{2f^2(\mathbf{C})} \quad (2.12)$$

with

$$f(\mathbf{C}) = 1 + \frac{3g^2}{4L^3|\mathbf{C}|^3} + \dots \quad (2.13)$$

If higher Fourier modes are taken into account, we obtain in the exact analogy with Eq. (1.2)

$$f(\mathbf{C}) = 1 + \frac{3g^2}{4} \sum_{n_k} \frac{1}{[\sum_k (C_k L_k - 2\pi n_k)^2]^{3/2}} + \dots \quad (2.14)$$

where, bearing in mind further applications, we assumed that the sizes of the torus  $L_k$ ,  $k = 1, 2, 3$ , do not coincide. The sum is divergent at large  $n_k$ . The coefficient of the logarithm gives the  $\beta$  function of the  $\mathcal{N} = 1$  SYM theory. For sure, this could be expected in advance. What is not quite trivial, however, is that the calculation in the truncated theory is *absolutely* parallel to the well-known calculation in 4 dimensions [8]: in four dimensions the corrections to the effective action are also given by the graphs in Fig. 1, and the gauge boson and the fermion contributions in the  $\beta$  function have the respective coefficients<sup>5</sup> 4 : -1. We will see soon that this is specific for supersymmetric theories. In nonsupersymmetric case, the  $\beta$  function can also be calculated with the chopoff technique, but the relevant graphs are different.

The bosonic effective action (2.12) can be supersymmetrized using the superfield technique developped in Ref. [10]. The explicit expression in

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<sup>5</sup>A remarkable fact is that one obtains the same ratio calculating the effective action in the instanton background field: the correct coefficient 6 in the  $\beta$  function is obtained as 8 - 2, where “8” is the number of bosonic zero modes and “2” is a half of the number of fermionic zero modes in the instanton background.

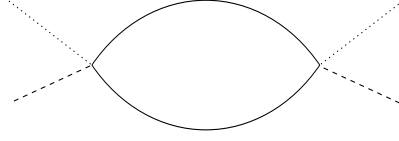


Figure 2: Renormalization of the kinetic term in scalar QED in the QM limit. The dashed lines correspond to  $\dot{\mathbf{A}}$  and the pointed lines — to  $\mathbf{A}$ .

components was written in Ref. [5]:

$$\begin{aligned} \frac{g^2}{L_1 L_2 L_3} \mathcal{L} = & \frac{1}{2f^2} \dot{C}^j \dot{C}^j + \frac{i}{2f^2} (\bar{\Psi} \dot{\Psi} - \dot{\bar{\Psi}} \Psi) - \frac{\partial_i f}{f^3} \epsilon^{ijk} \dot{C}_j \bar{\Psi} \sigma_k \Psi + \\ & + \frac{D^2}{2f^2} + \frac{D \partial_i f}{f^3} \bar{\Psi} \sigma_i \Psi - \frac{1}{8} \partial^2 \left( \frac{1}{f^2} \right) (\bar{\Psi})^2 (\Psi)^2, \end{aligned} \quad (2.15)$$

where  $\Psi = \psi f$ ,  $\psi$  and  $\bar{\psi}$  being the canonically conjugated variables of Eq. (1.1);  $D$  is the auxiliary field. The action corresponding to the Lagrangian (2.15) is invariant under the transformations

$$\begin{aligned} \delta_\epsilon C_k &= \bar{\epsilon} \sigma_k \Psi + \bar{\Psi} \sigma_k \epsilon, \\ \delta_\epsilon \Psi_\alpha &= -i (\sigma_k \epsilon)_\alpha \dot{C}_k + \epsilon_\alpha D, \\ \delta_\epsilon D &= i (\dot{\bar{\Psi}} \epsilon - \bar{\epsilon} \dot{\Psi}). \end{aligned} \quad (2.16)$$

The chopoff procedure works also for nonsupersymmetric theories. Consider the simplest case of scalar QED. The one-loop correction to the effective Lagrangian is just

$$\delta \mathcal{L}_{\text{scalar QED}} = -i \log \det(-\partial_0^2 - A_k^2). \quad (2.17)$$

The double derivative term is given by the graph depicted in Fig. 2. We obtain

$$\begin{aligned} \delta \mathcal{L}_{\text{scalar QED}} = & -(\mathbf{A} \dot{\mathbf{A}})^2 \frac{\partial^2}{\partial \epsilon^2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{[\omega^2 + \mathbf{A}^2][(\omega + \epsilon)^2 + \mathbf{A}^2]} \Big|_{\epsilon=0} \\ & = \frac{(\mathbf{A} \dot{\mathbf{A}})^2}{8|\mathbf{A}|^5}. \end{aligned} \quad (2.18)$$



Restoring the contribution of the higher Fourier modes,  $\mathbf{A} \rightarrow \mathbf{A} - 2\pi\mathbf{n}/L$ , and performing the summation over  $\mathbf{n}$ , with averaging over directions  $n_j n_k \equiv \frac{1}{3}\mathbf{n}^2\delta_{jk}$ , we reproduce the known result for the one-loop  $\beta$  function in the scalar QED,

$$\left. \frac{1}{e^2(L)} \right|_{\text{scalar QED}} = \frac{1}{e_0^2} + \frac{1}{24\pi^2} \ln(\Lambda_{UV} L) . \quad (2.19)$$

Therefore, the chopoff method works also for nonsupersymmetric theories. Actually, it was applied before to pure Yang–Mills theory in Ref. [11]. There are three important distinctions, however:

- In the nonsupersymmetric case, the effective Lagrangian calculated in the Abelian background (2.6) involves besides the kinetic term also the potential part  $\propto |\mathbf{C}|$ . As a result, in the non-Abelian case, the true slow variables are not only Abelian, but all zero Fourier modes of the gauge potential, and the Born–Oppenheimer Lagrangian becomes more complicated.
- A remark related to the previous one is that the metric in Eq. (2.18) is not conformally flat as it is in Eq. (2.12).
- The graph in Fig. 2 which determines the correction to the effective Lagrangian in the QM limit is quite different from the standard graph giving the  $\beta$  function in four dimensions. In particular, the former does not diverge in the ultraviolet in four dimensions.

### 3 Nonrenormalisation theorems.

In the dimensionally reduced  $\mathcal{N} = 4$  SYM theory (alias, maximally supersymmetric quantum mechanics, alias matrix model, alias the system of  $D0$ -branes), the corrections to the metric on the moduli space are absent. There are  $D$ -brane arguments in favor of this conclusion [12], it was confirmed by explicit calculation [6], and finally *proven* using simple symmetry arguments [13]. To make the paper self-contained, we present here a somewhat refined version of these arguments.

In the maximally supersymmetric  $SU(2)$  theory the effective Lagrangian is written in terms of a 9-dimensional vector  $C_k$  and a *real* 16-component

$SO(9)$  spinor  $\lambda_\alpha$ . The Lagrangian must be invariant with respect to the supersymmetry transformations

$$\begin{aligned}\delta_\epsilon C_k &= -i \epsilon \gamma_k \lambda , \\ \delta_\epsilon \lambda_\alpha &= \left( \gamma_k \dot{C}_k \epsilon \right)_\alpha + [M(c) \epsilon]_\alpha ,\end{aligned}\tag{3.20}$$

where  $\epsilon$  is a real Grassmann spinor and  $\gamma_k$  are the 9-dimensional  $\gamma$  matrices,  $\gamma_j \gamma_k + \gamma_k \gamma_j = 2\delta_{jk}$ . They are all real and symmetric. The transformations (3.20) represent an analog of (2.16) with the auxiliary field expressed out.

The commutator of two SUSY transformations with parameters  $\epsilon_1$  and  $\epsilon_2$  should amount to a time translation. A trivial calculation gives

$$[\delta_1, \delta_2] C_k = -2i \epsilon_2 \epsilon_1 \dot{C}_k - i \epsilon_2 \{ \gamma_k M + M^T \gamma_k \} \epsilon_1 ,\tag{3.21}$$

and we conclude that

$$\gamma_k M + M^T \gamma_k = 0 .\tag{3.22}$$

As was noticed in Ref.[13], this implies that  $M = 0$ . Let us prove it. On the first step, note that any  $M$  satisfying (3.22) commutes with all generators  $J_{kj} = \frac{1}{4}[\gamma_k, \gamma_j]$  of  $Spin(9)$ . This means that, for any set  $\lambda$  belonging to the spinor representation of  $Spin(9)$ , the set  $M\lambda$  also forms a spinor representation. Hence  $M = \xi R$ , where  $\xi$  is a real number and  $R \in Spin(9)$ . But  $R$  commutes with all generators of  $Spin(9)$  and should belong to the center of  $Spin(9)$ , i.e.  $M = \pm \xi I$ . Then (3.22) tells us that  $\xi = 0$ .

When proving this, we used implicitly the fact that the real spinor representation of  $SO(9)$  is *irreducible*. If it could be decomposed in a direct sum of two other representations, we could choose  $M$  as  $\text{diag}(\xi_1 z_1, \xi_2 z_2)$ , with  $z_1$  and  $z_2$  belonging to the center of the group in the corresponding subspaces. Such  $M$  would not be necessarily proportional to  $I$ . This discussion is not purely academic. Actually, for  $Spin(3)$  and  $Spin(5)$ , where real spinor representations are reducible, nontrivial matrices satisfying (3.22) *exist*.

The only structure not involving higher derivatives<sup>6</sup> and invariant with respect to the transformations (3.20) with  $M = 0$  is

$$\frac{1}{2} \left[ \dot{C}_k^2 + i \lambda \dot{\lambda} \right] .$$

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<sup>6</sup>Higher derivative terms, in particular the term  $\propto (\dot{C}_k \dot{C}_k)^2$  and its superpartners are allowed. See [6, 14] for detailed discussion.

Nontrivial corrections to the metric are not allowed. Bearing in mind the discussion in the previous section, this simultaneously proves that the  $\beta$  function in  $\mathcal{N} = 4$  SYM theory vanishes exactly in all loops.

In the  $\mathcal{N} = 2$  case, the corrections to the metric survive, but the presence of 4 different complex supercharges dictates that the function  $f^{-2}(\mathbf{C})$  ( $\mathbf{C}$  is now a 5-dimensional vector. In four dimensions this corresponds to the gauge potential and a complex scalar.) is not arbitrary, but should be a harmonic function,  $\Delta^{(5)} f^{-2}(\mathbf{C}) = 0$  [15, 16]. The  $O(5)$  invariance, which is manifest in the chopoff quantum-mechanical limit, tells us then that the only allowed form of the effective Lagrangian is

$$\mathcal{L}^{\text{eff}} = \frac{\dot{\mathbf{C}}^2}{2} \left( 1 + \frac{\text{const}}{|\mathbf{C}|^3} \right) , \quad (3.23)$$

i.e. all the corrections beyond one loop vanish. But this means also that multiloop corrections to the  $\beta$  function vanish in this case.

## 4 Effective action in (1+1) dimensions.

Consider the gauge SYM theory with  $SU(2)$  gauge group compactified on  $T^2$  rather than on  $T^3$ . The low-energy dynamics is described by an effective (1+1) dimensional field theory depending on *two* bosonic variables  $C_{1,2}(z, t)$  and their fermionic superpartners. The effective Lagrangian represents in this case a supersymmetric Kählerian  $\sigma$ -model [17]. The corresponding Kählerian manifold represents a 2-torus dual to the spatial torus. The precise form of the metric can be determined by a simple one-loop calculation. It can be done along the same lines as in Sect. 2. Calculating the graphs in Fig. 1, we obtain for the correction to the effective Lagrangian density

$$\Delta \mathcal{L}_{\text{bos}}^{d=2} = -3(\partial_\alpha C_j)^2 \int_{-\infty}^{\infty} \frac{d^2 p}{(2\pi)^2} \frac{1}{(p^2 + C_j^2)^2} = \frac{(\partial_\alpha C_j)^2}{4\pi C_j^2} , \quad (4.24)$$

where  $\alpha = 0, 3$  and  $j = 1, 2$ . We obtain

$$\frac{g^2}{L^2} \mathcal{L}_{\text{bos}}^{d=2} = \frac{1}{2} (\partial_\alpha C_j)^2 \tilde{f}^{-2}(C_j) , \quad (4.25)$$

where

$$\tilde{f}(C_j) = 1 + \frac{3g^2}{4\pi L^2 C_j^2} . \quad (4.26)$$

The fermion terms can be restored by supersymmetrizing. The full effective Lagrangian is

$$g^2 \mathcal{L}^{d=2} = \int d^2\theta d^2\bar{\theta} \left[ \bar{\Phi}\Phi - \frac{3}{4\pi} \ln \Phi \ln \bar{\Phi} \right], \quad (4.27)$$

where  $\Phi$  is a chiral superfield with the lowest component  $\phi = L(C_1 + iC_2)/\sqrt{2}$ . When deriving (4.25), we neglected higher Fourier modes associated with compactified directions. This is justified if  $|\phi| \ll 1$ . (On the other hand, we have to keep  $|\phi| \gg g$ , otherwise higher loop corrections become relevant.) For  $|\phi| \sim 1$ , the higher Fourier modes should be taken into account, which can be easily done by substituting  $C_j \rightarrow C_j - 2\pi n_j/L$  and performing the sum over integer 2-dimensional  $n_j$ .

It is very instructive to explore the relationship between the effective  $d = 2$  Lagrangian (4.27) and our nonstandard (0+1)-dimensional  $\sigma$  model (2.15). To begin with, let us study the geometric structure of the Lagrangian (2.15). To this end, we integrate out the auxiliary field  $D$  and express the Lagrangian in the form

$$\begin{aligned} \frac{g^2}{L_1 L_2 L_3} \mathcal{L} = & \frac{1}{2} g_{jk} \dot{C}^j \dot{C}^k + \frac{i}{2} (\bar{\psi} \dot{\psi} - \dot{\bar{\psi}} \psi) + i \omega_j^{ab} \dot{C}^j \bar{\psi} \sigma^{ab} \psi + \\ & + \frac{1}{4} [f(\partial^2 f) - 2(\partial_j f)^2] (\bar{\psi})^2 (\psi)^2, \end{aligned} \quad (4.28)$$

where we have raised the index of the vector  $C^j$  indicating its contravariant nature,  $g_{jk} = f^{-2} \delta_{jk}$ ,  $\psi = \Psi/f$ ,  $\sigma^{ab} = \frac{i}{2} \epsilon^{abc} \sigma^c$  is the generator of rotations in the tangent space, and

$$\omega_i^{ab} = \delta_i^b \partial^a \log(f) - \delta_i^a \partial^b \log(f) \quad (4.29)$$

is the spin connection on a conformally flat manifold with the natural choice of the dreibein,  $e_j^a = f^{-1} \delta_j^a$ .

We have succeeded in presenting the bifermion term in a nice geometric form. However, the 4-fermion term in Eq. (4.28) does not have an obvious geometric interpretation. In particular, its coefficient *is not* a 3-dimensional scalar curvature.

To establish the relation of (4.28) to the Kählerian model, let us consider the original theory on an *asymmetric* torus  $L_3 \gg L_1 = L_2 \equiv L$ . The effective QM model is given by the Lagrangian (4.25) with  $f(C_{1,2}, C_3)$  written in Eq.

(2.14). If  $L_3$  is very large, the range where  $C_3$  changes is very small. In the limit  $L_3 \rightarrow \infty$ ,  $C_3$  is frozen to zero, but to perform this limit, we cannot just set  $C_3 = 0$  in Eq. (2.14), but rather *average* over  $C_3$  within the range  $0 \leq C_3 \leq 2\pi/L_3$  and simultaneously perform the summation over  $n_3$ . This amounts to calculating the integral

$$\frac{1}{f^2(C_3, C_{1,2})} \rightarrow 1 - \frac{3g^2}{2} \sum_{n_j} \int \frac{da}{2\pi} \frac{1}{[\sum_{j=1,2}(LC_j - 2\pi n_j)^2 + a^2]^{3/2}} ,$$

which exactly gives the correction to the metric in Eq. (4.25), (4.26).

The Lagrangian (4.28) involves also the kinetic term for the field  $C_3$  and the term where  $\dot{C}_3$  multiplies a bifermion structure. Performing the functional integral of  $\exp\{iS\}$  over  $\prod_t dC_3(t)$ , we arrive at the expression

$$\mathcal{L} = \frac{1}{2} \tilde{g}_{jk} \dot{y}^j \dot{y}^k + i \bar{\chi}^a \left( \delta^{ab} \partial_t - \omega_j^{ab} \dot{y}^j \right) \chi^b + \frac{1}{8} R (\bar{\chi})^2 (\chi)^2 , \quad (4.30)$$

where  $y^j = Lc^j/g$ ,  $\tilde{g}_{jk} = \tilde{f}^{-2} \delta_{jk}$ ,  $\chi$  is related to  $\psi$  by a unitary transformation,  $\omega_j^{ab}$  is the two-dimensional spin connection (4.29), and  $R = 2[\tilde{f} \partial^2 \tilde{f} - (\partial_j \tilde{f})^2]$  is the two-dimensional scalar curvature. The Lagrangian (4.30) coincides with the standard Lagrangian of the supersymmetric  $\sigma$  model [18]. in the QM limit. The  $(1+1)$  effective Lagrangian could be obtained if taking into account the higher Fourier harmonics  $\propto \exp\{izn/L_3\}$  of  $C_j(z, t)$  and  $\psi(z, t)$  in the background.

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## References

- [1] E. Witten, J. High Energy Phys. **9802** (1998) 006; A. Keurentjes, A. Rosly, and A.V. Smilga, Phys. Rev. **58** (1998) 081701; V.G. Kac and

- A.V. Smilga, hep-th/9902029, published in *The Many Faces of the Superworld* (World Scientific, 2000), ed. M.A. Shifman; A. Keurentjes, J. High Energy Phys. **9905** (1999) 001, 014.
- [2] A.V. Smilga, JETP **64** (1986) 8; B.Yu. Blok, A.V. Smilga, Nucl. Phys. **B287** (1987) 589.
  - [3] A.V. Smilga, Nucl. Phys. **B291** (1987) 241.
  - [4] J. Polchinski, "TASI Lectures on D-branes", hep-th/9611050.
  - [5] A.V. Smilga, hep-th/0201048.
  - [6] K. Becker and M. Becker, Nucl. Phys. **B506** (1997) 48.
  - [7] M. Douglas, J. Polchinski and A. Strominger, JHEP 9712 (1997) 003; E. Akhmedov, Phys. Rev. **D59** (1999) 101901, hep-th/9812038.
  - [8] L.F. Abbot, Nucl. Phys. **B185** (1981) 189; M. Peskin, *An Introduction in Quantum Field Theory*, Addison-Wesley, 1995, p. 533.
  - [9] V.A. Novikov et al, Phys. Lett. **B217** (1989) 103; for a modern review see M.A. Shifman and A.I. Vainshtein hep-th/9902018, in: M.A. Shifman, *ITEP Lectures on Particle Physics and Field Theory*, World Scientific, 1999, v.2, p.485.
  - [10] E.A. Ivanov and A.V. Smilga, Phys. Lett. **B257** (1991) 79.
  - [11] M. Lüscher, Nucl. Phys. **B219** (1983) 233.
  - [12] M.R. Douglas et al, Nucl. Phys. **B485** (1997) 85.
  - [13] S. Paban, S. Sethi, and M. Stern, Nucl. Phys. **B534** (1998) 137.
  - [14] K. Becker et al, Phys. Rev. **D56** (1997) 3174; Y. Okawa and T. Yoneya, Nucl. Phys. **B538** (1999) 67; R. Helling et al, Nucl. Phys. **B559** (1999) 184.
  - [15] D.-E. Diaconescu and R. Entin, Phys. Rev. **D56** (1997) 8045.
  - [16] B. Zupnik, Theor. Math. Phys. **120** (1999) 365.

- [17] This is kind of folklore, but we did not see in the literature any explicit calculation.
- [18] D.Z. Freeman and P.K. Townsend, Nucl. Phys. **B250** (1985) 689.